Computers and Mathematics with Applications 58 (2009) 1383–1390

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Generalizations of $(\in, \in \vee \mathrm{q})$-fuzzy subalgebras in BCK/BCI-algebras

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A R T I C L E   I N F O

Article history:
Received 19 August 2008
Received in revised form 1 July 2009
Accepted 10 July 2009

Keywords:
BCK/BCI-algebra
$(\in, \mathrm{q}_k)$-fuzzy subalgebra
$(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra
$(\in \vee \mathrm{q}_k)$-level subalgebra

A B S T R A C T

More general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set is considered, and generalizations of results in the papers [Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc. 42 (4) (2005) 703–711; Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras, Kyungpook Math. J. 47 (2007) 403–410] are discussed. The notions of $(\in, \mathrm{q}_k)$-fuzzy subalgebras and $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebras in a BCK/BCI-algebra X are introduced, and several properties are investigated. Characterizations of $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra in a BCK/BCI-algebra X are discussed.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which have been pointed out in [3]. Maji et al. [4] and Molodtsov [3] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [3] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, research on the soft set theory is progressing rapidly. Maji et al. [4] described the application of soft set theory to a decision making problem. They also studied several operations on the theory of soft sets (see [5]). Chen et al. [6] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The present author [7] (together with his colleagues [8]) applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras and d-algebras, and introduced the notions of soft BCK/BCI-algebras, soft subalgebras and soft d-algebras, and then investigated their basic properties. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [9]. The fuzzy set theory is applied to BCK-algebras in [10–15]. Murali [16] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [17], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [18,19] initiated the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee \mathrm{q})$-fuzzy subgroup. In particular, an $(\in, \in \vee \mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of

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doi:10.1016/j.camwa.2009.07.043

the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song [20] discussed general forms of fuzzy interior ideals in semigroups. Also, Jun [1,2] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated the related results. Ma et al. [21] considered $(\in, \in\vee\,\mathrm{q})$-interval-valued fuzzy ideals of BCI-algebras.

In this paper, we consider more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set, and then we deal with generalizations of results which are obtained in the papers [1,2]. As a generalization of $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebras, we introduce the notions of $(\in, \mathrm{q}_k)$-fuzzy subalgebras and $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebras in a BCK/BCI-algebra X, and investigate several properties. We also characterize the $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebra in a BCK/BCI-algebra X which is a generalization of $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra. Based on this article, we will try to study the generalization of the (α, β) types of ideals in BCK/BCI-algebras, and further generalize the generalized fuzzy interior ideals in semigroups. Moreover, We will discuss (α, β)-intuitionistic fuzzy theory in BCK/BCI-algebras, BL-algebras, MV-algebras etc., and study soft set theory in BCK/BCI-algebras by using the general form of (α, β)-type theory in BCK/BCI-algebras.

2. Preliminaries

By a *BCI-algebra*, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(ii) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(iii) $(\forall x \in X)\ (x * x = 0)$,
(iv) $(\forall x, y \in X)\ (x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. Huang and Jun [22] studied ideals and subalgebras in BCI-algebras. In what follows, X is a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [23,24] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy point (x, t) and a fuzzy set μ in a set X, Pu and Liu [17] introduced the symbol $(x, t)\alpha\mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$. To say that $(x, t) \in \mu$ (resp. $(x, t)\mathrm{q}\mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, (x, t) is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $(x, t) \in\vee\,\mathrm{q}\,\mu$ (resp. $(x, t) \in\wedge\,\mathrm{q}\,\mu$), we mean $(x, t) \in \mu$ or $(x, t)\mathrm{q}\mu$ (resp. $(x, t) \in \mu$ and $(x, t)\mathrm{q}\mu$). For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$.

3. Generalizations of $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebras

In what follows let α and β denote any one of $\in$, q, $\in\vee\,\mathrm{q}$, or $\in\wedge\,\mathrm{q}$ unless otherwise specified. To say that $(x, t)\overline{\alpha}\mu$, we mean $(x, t)\alpha\mu$ does not hold. Let X denote a BCK/BCI-algebras unless otherwise specified.

Definition 3.1 (*[1]*). A fuzzy set μ in X is said to be an (α, β)-*fuzzy subalgebra* of X, where $\alpha \neq\ \in\wedge\,\mathrm{q}$, if it satisfies:

$$(x, t_1)\alpha\mu,\ (y, t_2)\alpha\mu \Rightarrow (x * y, t_1 \wedge t_2)\beta\mu \tag{3.1}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

Let k denote an arbitrary element of $[0, 1)$ unless otherwise specified. To say that $(x, t)\ \mathrm{q}_k\,\mu$, we mean $\mu(x) + t + k > 1$. To say that $(x, t) \in\vee\,\mathrm{q}_k\,\mu$, we mean $(x, t) \in \mu$ or $(x, t)\ \mathrm{q}_k\,\mu$.

Definition 3.2. A fuzzy set μ in X is called an $(\in, \in\vee\,\mathrm{q}_k)$-*fuzzy subalgebra* of X if it satisfies:

$$(x, t_1) \in \mu,\ (y, t_2) \in \mu \Rightarrow (x * y, t_1 \wedge t_2) \in\vee\,\mathrm{q}_k\,\mu \tag{3.2}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

We give characterizations of an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebra.

Theorem 3.3. *A fuzzy set μ in X is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\}\right). \tag{3.3}$$

Proof. Let μ be an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebra of X. Assume that (3.3) is not valid. Then there exist $a, b \in X$ such that

$$\mu(a * b) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

If $\mu(a) \wedge \mu(b) < \frac{1-k}{2}$, then $\mu(a * b) < \mu(a) \wedge \mu(b)$. Hence

$$\mu(a * b) < t \leq \mu(a) \wedge \mu(b)$$

for some $t \in (0, 1)$. It follows that $(a, t) \in \mu$ and $(b, t) \in \mu$, but $(a * b, t) \overline{\in} \mu$. Moreover, $\mu(a * b) + t < 2t < 1 - k$, and so $(a * b, t)\, \overline{q_k}\, \mu$. Consequently $(a * b, t)\, \overline{\in \vee q_k}\, \mu$, this is a contradiction. If $\mu(a) \wedge \mu(b) \geq \frac{1-k}{2}$, then $\mu(a) \geq \frac{1-k}{2}$, $\mu(b) \geq \frac{1-k}{2}$ and $\mu(a * b) < \frac{1-k}{2}$. Thus $(a, \frac{1-k}{2}) \in \mu$ and $(b, \frac{1-k}{2}) \in \mu$, but $(a * b, \frac{1-k}{2}) \overline{\in} \mu$. Also,

$$\mu(a * b) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(a * b, \frac{1-k}{2})\, \overline{q_k}\, \mu$. Hence $(a * b, \frac{1-k}{2})\, \overline{\in \vee q_k}\, \mu$, again, a contradiction. Therefore (3.3) is valid.

Conversely, suppose that μ satisfies (3.3). Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{t_1, t_2, \frac{1-k}{2}\right\}.$$

Assume that $t_1 \leq \frac{1-k}{2}$ or $t_2 \leq \frac{1-k}{2}$. Then $\mu(x * y) \geq t_1 \wedge t_2$, which implies that $(x * y, t_1 \wedge t_2) \in \mu$. Now, suppose that $t_1 > \frac{1-k}{2}$ and $t_2 > \frac{1-k}{2}$. Then $\mu(x * y) \geq \frac{1-k}{2}$, and thus

$$\mu(x * y) + t_1 \wedge t_2 > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(x * y, t_1 \wedge t_2)\, q_k\, \mu$. Hence $(x * y, t_1 \wedge t_2) \in\vee q_k\, \mu$, and consequently, μ is an $(\in, \in\vee q_k)$-fuzzy subalgebra of X. $\square$

Corollary 3.4 ([1]). *A fuzzy set μ in X is an $(\in, \in\vee q)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}). \tag{3.4}$$

Proof. It follows from taking $k = 0$ in Theorem 3.3. $\square$

Theorem 3.5. *Let μ be a fuzzy set in X. Then μ is an $(\in, \in\vee q_k)$-fuzzy subalgebra of X if and only if the level subset*

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra of X for all $t \in (0, \frac{1-k}{2}]$.

Proof. Assume that μ is an $(\in, \in\vee q_k)$-fuzzy subalgebra of X. Let $t \in (0, \frac{1-k}{2}]$ and $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from (3.3) that

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq t \wedge \frac{1-k}{2} = t$$

so that $x * y \in U(\mu; t)$. Hence $U(\mu; t)$ is a subalgebra of X.

Conversely, suppose that $U(\mu; t)$ is a subalgebra of X for all $t \in (0, \frac{1-k}{2}]$. If (3.3) is not valid, then there exist $a, b \in X$ such that

$$\mu(a * b) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(a * b) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Then $t \in (0, \frac{1-k}{2}]$ and $a, b \in U(\mu; t)$. Since $U(\mu; t)$ is a subalgebra of X, it follows that $a * b \in U(\mu; t)$ so that $\mu(a * b) \geq t$. This is a contradiction. Therefore (3.3) is valid, and μ is an $(\in, \in\vee q_k)$-fuzzy subalgebra of X by Theorem 3.3. $\square$

Taking $k = 0$ in Theorem 3.5, we have the following corollary.

Corollary 3.6 ([2]). *Let μ be a fuzzy set in X. Then μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X if and only if the level subset*

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra of X for all $t \in (0, 0.5]$.

Example 3.7. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table (see [25]):

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy set in X defined by $\mu(0) = 0.6$, $\mu(a) = 0.7$, and $\mu(b) = \mu(c) = 0.3$.

(1) If $k = 0.4$, then $U(\mu; t) = X$ for all $t \in (0, 0.3]$. Hence μ is an $(\in, \in\vee\, \mathrm{q}_{0.4})$-fuzzy subalgebra of X.

(2) If $k = 0.2$, then

$$U(\mu; t) = \begin{cases} X & \text{if } t \in (0, 0.3], \\ \{0, a\} & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since X and $\{0, a\}$ are subalgebras of X, μ is an $(\in, \in\vee\, \mathrm{q}_{0.2})$-fuzzy subalgebra of X.

Example 3.8. Let X be the BCI-algebra given in Example 3.7. Let μ be a fuzzy set in X defined by $\mu(0) = 0.42$, $\mu(a) = \mu(c) = 0.4$, and $\mu(b) = 0.48$. If $k = 0.04$, then

$$U(\mu; t) = \begin{cases} X & \text{if } t \in (0, 0.4], \\ \{0, b\} & \text{if } t \in (0.4, 0.42], \\ \{b\} & \text{if } t \in (0.42, 0.48]. \end{cases}$$

Note that $U(\mu; t)$ is not a subalgebra for $t \in (0.42, 0.48]$. Hence μ is not an $(\in, \in\vee\, \mathrm{q}_{0.04})$-fuzzy subalgebra of X.

Theorem 3.9. *Every $(\in, \in)$-fuzzy subalgebra of X is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra X.*

Proof. Straightforward. □

Taking $k = 0$ in Theorem 3.9, we have the following corollary.

Corollary 3.10 ([1]). *Every $(\in, \in)$-fuzzy subalgebra of X is an $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebra X.*

The converse of Theorem 3.9 is not true as seen in the following example.

Example 3.11. Consider the $(\in, \in\vee\, \mathrm{q}_{0.4})$-fuzzy subalgebra of X which is given in Example 3.7. Then μ is not an $(\in, \in)$-fuzzy subalgebra of X since $(a, 0.62) \in \mu$ and $(a, 0.66) \in \mu$, but $(a * a, 0.62 \wedge 0.66) = (0, 0.62)\, \overline{\in}\, \mu$.

Definition 3.12. A fuzzy set μ in X is called an $(\in, \mathrm{q}_k)$-*fuzzy subalgebra* of X if it satisfies:

$$(x, t_1) \in \mu,\ (y, t_2) \in \mu \ \Rightarrow\ (x * y, t_1 \wedge t_2)\, \mathrm{q}_k\, \mu \tag{3.5}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

Theorem 3.13. *Every $(\in, \mathrm{q}_k)$-fuzzy subalgebra of X is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra X.*

Proof. Straightforward. □

Taking $k = 0$ in Theorem 3.13, we have the following corollary.

Corollary 3.14. *Every $(\in, \mathrm{q})$-fuzzy subalgebra of X is an $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebra X.*

The following example shows that the converse of Theorem 3.13 does not hold.

Example 3.15. Consider the $(\in, \in\vee\, \mathrm{q}_{0.2})$-fuzzy subalgebra of X which is given in Example 3.7. Note that $(a, 0.4) \in \mu$ and $(b, 0.25) \in \mu$, but $(a * c, 0.4 \wedge 0.25) = (b, 0.25)\, \overline{\mathrm{q}_{0.2}}\, \mu$ since $\mu(b) + 0.25 + 0.2 < 1$. Hence μ is not an $(\in, \mathrm{q}_{0.2})$-fuzzy subalgebra of X.

Theorem 3.16. *Let X be a BCK/BCI-algebra. If $0 \le k < r < 1$, then every $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X is an $(\in, \in\vee\, \mathrm{q}_r)$-fuzzy subalgebra of X.*

Proof. Straightforward. □

The following example shows that if $0 \le k < r < 1$, then an $(\in, \in\vee\, \mathrm{q}_r)$-fuzzy subalgebra of X may not be an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X.

Example 3.17. Let X and μ be as in Example 3.8. If $r = 0.16$, then

$$U(\mu; t) = \begin{cases} X & \text{if } t \in (0, 0.4], \\ \{0, b\} & \text{if } t \in (0.4, 0.42]. \end{cases}$$

Since X and $\{0, b\}$ are subalgebras of X, we know that μ is an $(\in, \in \vee \mathrm{q}_{0.16})$-fuzzy subalgebra of X by Theorem 3.5. But μ is not an $(\in, \in \vee \mathrm{q}_{0.04})$-fuzzy subalgebra of X (see Example 3.8).

Let S be a subset of X. Consider a fuzzy set μ_S in X defined by

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$.

Theorem 3.18. *A nonempty subset S of X is a subalgebra of X if and only if the fuzzy set μ_S in X is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X.*

Proof. Let S be a subalgebra of X. Then $U(\mu_S; t)$ is clearly a subalgebra of X for all $t \in (0, \frac{1-k}{2}]$. Hence μ_S is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X by Theorem 3.5.

Conversely, assume that μ_S is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X. Let $x, y \in S$. Then

$$\mu_S(x * y) \geq \min\left\{\mu_S(x), \mu_S(y), \frac{1-k}{2}\right\} = 1 \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, $\mu_S(x * y) = 1$ and so $x * y \in S$. Hence S is a subalgebra of X. □

Theorem 3.19. *Let S be a subalgebra of X. For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra μ of X such that $U(\mu; t) = S$.*

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$, where $t \in (0, \frac{1-k}{2}]$. Obviously, $U(\mu; t) = S$. Assume that

$$\mu(a * b) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}$$

for some $a, b \in X$. Since $\#\mathrm{Im}(\mu) = 2$, it follows that $\mu(a * b) = 0$ and $\min\{\mu(a), \mu(b), \frac{1-k}{2}\} = t$. Hence $\mu(a) = t = \mu(b)$, and so $a, b \in S$. Since S is a subalgebra of X, $a * b \in S$. Thus $\mu(a * b) = t$, which is a contradiction. Therefore

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\}$$

for all $x, y \in X$. Using Theorem 3.3, we know that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X. □

Taking $k = 0$ in Theorem 3.19, we have the following corollary.

Corollary 3.20 ([2]). *Let S be a subalgebra of X. For every $t \in (0, 0.5]$, there exists an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra μ of X such that $U(\mu; t) = S$.*

Theorem 3.21. *Let μ be an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X such that $\mu(x) < \frac{1-k}{2}$ for all $x \in X$. Then μ is an $(\in, \in)$-fuzzy subalgebra of X.*

Proof. It is straightforward by using (3.3). □

If we take $k = 0$ in Theorem 3.21, then we have the following corollary.

Corollary 3.22 ([2]). *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X such that $\mu(x) < 0.5$ for all $x \in X$. Then μ is an $(\in, \in)$-fuzzy subalgebra of X.*

Theorem 3.23. *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebras of X. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Assume that $(x * y, t_1 \wedge t_2) \overline{\in \vee \mathrm{q}_k}\, \mu$. Then $\mu(x * y) < t_1 \wedge t_2$ and $\mu(x * y) + t_1 \wedge t_2 \leq 1 - k$, which imply that

$$\mu(x * y) < \frac{1-k}{2}. \tag{3.6}$$

Let $\Omega_1 := \{i \in \Lambda \mid (x*y, t_1 \wedge t_2) \in \mu_i\}$ and

$$\Omega_2 := \{i \in \Lambda \mid (x*y, t_1 \wedge t_2)\, \mathrm{q}_k\, \mu_i\} \cap \{j \in \Lambda \mid (x*y, t_1 \wedge t_2) \overline{\in} \mu_j\}.$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then $(x*y, t_1 \wedge t_2) \in \mu_i$ for all $i \in \Lambda$, that is, $\mu_i(x*y) \geq t_1 \wedge t_2$ for all $i \in \Lambda$, which yields $\mu(x*y) \geq t_1 \wedge t_2$. This is a contradiction. Hence $\Omega_2 \neq \emptyset$, and so for every $i \in \Omega_2$ we have $\mu_i(x*y) < t_1 \wedge t_2$ and $\mu_i(x*y) + t_1 \wedge t_2 > 1-k$. It follows that $t_1 \wedge t_2 > \frac{1-k}{2}$. Now $(x, t_1) \in \mu$ implies $\mu(x) \geq t_1$ and thus $\mu_i(x) \geq \mu(x) \geq t_1 \geq t_1 \wedge t_2 > \frac{1-k}{2}$ for all $i \in \Lambda$. Similarly $\mu_i(y) > \frac{1-k}{2}$ for all $i \in \Lambda$. Next suppose that $t := \mu_i(x*y) < \frac{1-k}{2}$. Taking $t < r < \frac{1-k}{2}$, we get $(x, r) \in \mu_i$ and $(y, r) \in \mu_i$, but $(x*y, r \wedge r) = (x*y, r) \overline{\in \vee\, \mathrm{q}_k}\, \mu_i$. This contradicts that μ_i is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X. Hence $\mu_i(x*y) \geq \frac{1-k}{2}$ for all $i \in \Lambda$, and so $\mu(x*y) \geq \frac{1-k}{2}$ which contradicts (3.6). Therefore $(x*y, t_1 \wedge t_2) \in\vee\, \mathrm{q}_k\, \mu$ and consequently μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X. □

Taking $k = 0$ in Theorem 3.23, we have the following corollary.

Corollary 3.24 (*[1]*). *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebras of X. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebra of X.*

The following example shows that there exists $k \in [0, 1)$ such that the union of two $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebras of X may not be an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X.

Example 3.25. Let $X = \{0, a, b, c\}$ be a BCI-algebras which is given in Example 3.7 and let μ be an $(\in, \in\vee\, \mathrm{q}_{0.2})$-fuzzy subalgebra of X which is described in Example 3.7(2). Let ν be a fuzzy set in X defined by $\nu(0) = 0.4$, $\nu(a) = \nu(c) = 0.3$, and $\nu(b) = 0.5$. Then

$$U(\nu; t) = \begin{cases} X & \text{if } t \in (0, 0.3], \\ \{0, b\} & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since X and $\{0, b\}$ are subalgebras of X, ν is an $(\in, \in\vee\, \mathrm{q}_{0.2})$-fuzzy subalgebra of X by Theorem 3.5. The union $\mu \cup \nu$ of μ and ν is given by $(\mu \cup \nu)(0) = 0.6$, $(\mu \cup \nu)(a) = 0.7$, $(\mu \cup \nu)(b) = 0.5$ and $(\mu \cup \nu)(c) = 0.3$. Hence

$$U(\mu \cup \nu; t) = \begin{cases} X & \text{if } t \in (0, 0.3], \\ \{0, a, b\} & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since $\{0, a, b\}$ is not a subalgebra of X, it follows from Theorem 3.5 that $\mu \cup \nu$ is not an $(\in, \in\vee\, \mathrm{q}_{0.2})$-fuzzy subalgebra of X.

For any fuzzy set μ in X and $t \in (0, 1]$, we denote

$$\mu_t := \{x \in X \mid (x, t)\, \mathrm{q}_k\, \mu\} \quad \text{and} \quad [\mu]_t := \{x \in X \mid (x, t) \in\vee\, \mathrm{q}_k\, \mu\}.$$

Obviously, $[\mu]_t = U(\mu; t) \cup \mu_t$.

Theorem 3.26. *Let μ be a fuzzy set in X. Then μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X if and only if $[\mu]_t$ is a subalgebra of X for all $t \in (0, 1]$.*

We call $[\mu]_t$ an $(\in\vee\, \mathrm{q}_k)$-*level subalgebra* of μ.

Proof. Assume that μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy subalgebra of X and let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $(x, t) \in\vee\, \mathrm{q}_k\, \mu$ and $(y, t) \in\vee\, \mathrm{q}_k\, \mu$, that is, $\mu(x) \geq t$ or $\mu(x) + t > 1-k$, and $\mu(y) \geq t$ or $\mu(y) + t > 1-k$. Using Theorem 3.3, we have $\mu(x*y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$.

Case 1. $\mu(x) \geq t$ and $\mu(y) \geq t$. If $t > \frac{1-k}{2}$, then

$$\mu(x*y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Hence $\mu(x*y) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1-k$, and so $(x*y, t)\, \mathrm{q}_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x*y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq t,$$

and thus $(x*y, t) \in \mu$. Therefore $(x*y, t) \in\vee\, \mathrm{q}_k\, \mu$, i.e., $x*y \in [\mu]_t$.

Case 2. $\mu(x) \geq t$ and $\mu(y) + t > 1-k$. If $t > \frac{1-k}{2}$, then

$$\mu(x*y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \mu(y) \wedge \frac{1-k}{2} > (1-k-t) \wedge \frac{1-k}{2} = 1-k-t,$$

and so $(x * y, t)\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{t, 1-k-t, \frac{1-k}{2}\right\} = t.$$

Hence $(x * y, t) \in \mu$, and thus $(x * y, t) \in \vee\, q_k\, \mu$, i.e., $x * y \in [\mu]_t$.

Case 3. $\mu(x) + t > 1 - k$ and $\mu(y) \geq t$. If $t > \frac{1-k}{2}$, then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \mu(x) \wedge \frac{1-k}{2} > (1-k-t) \wedge \frac{1-k}{2} = 1-k-t,$$

and so $(x * y, t)\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{1-k-t, t, \frac{1-k}{2}\right\} = t.$$

Hence $(x * y, t) \in \mu$, and thus $(x * y, t) \in \vee\, q_k\, \mu$, i.e., $x * y \in [\mu]_t$.

Case 4. $\mu(x) + t > 1 - k$ and $\mu(y) + t > 1 - k$. If $t > \frac{1-k}{2}$, then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} > (1-k-t) \wedge \frac{1-k}{2} = 1-k-t.$$

Thus $(x * y, t)\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x * y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq (1-k-t) \wedge \frac{1-k}{2} = \frac{1-k}{2} \geq t,$$

and therefore $(x * y, t) \in \mu$. Hence $(x * y, t) \in \vee\, q_k\, \mu$, i.e., $x * y \in [\mu]_t$. Consequently, $[\mu]_t$ is a subalgebra of X.

Conversely, let μ be a fuzzy set in X and $t \in (0, 1]$ be such that $[\mu]_t$ is a subalgebra of X. If it is possible, let

$$\mu(x * y) < t \leq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \tag{3.7}$$

for some $t \in (0, \frac{1-k}{2})$. Then $x, y \in U(\mu; t) \subseteq [\mu]_t$, which implies that $x*y \in [\mu]_t$. Hence $\mu(x*y) \geq t$ or $\mu(x*y)+t+k > 1$, a contradiction. Therefore $\mu(x * y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in X$. Using Theorem 3.3, we conclude that μ is an $(\in, \in \vee\, q_k)$-fuzzy subalgebra of X. $\square$

Note that Theorem 3.26 is a generalization of [2, Theorem 3.11]. A fuzzy set μ in X is said to be *proper* if $\mathrm{Im}(\mu)$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.27. *Let μ be an $(\in, \in \vee\, q_k)$-fuzzy subalgebra of X such that $\#\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in \vee\, q_k)$-fuzzy subalgebras of X such that μ can be expressed as the union of them.*

Proof. Let $\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\} = \{t_1, t_2, \ldots, t_r\}$, where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then the chain of $(\in \vee\, q_k)$-level subalgebras of μ is

$$[\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r} = X.$$

Let ν and γ be fuzzy sets in X defined by

$$\nu(x) = \begin{cases} t_1 & \text{if } x \in [\mu]_{t_1}, \\ t_2 & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{t_1}, \\ \cdots & \\ t_r & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}}, \end{cases}$$

and

$$\gamma(x) = \begin{cases} \mu(x) & \text{if } x \in [\mu]_{\frac{1-k}{2}}, \\ k & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{\frac{1-k}{2}}, \\ t_3 & \text{if } x \in [\mu]_{t_3} \setminus [\mu]_{t_2}, \\ \cdots & \\ t_r & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then ν and γ are $(\in, \in \vee\, q_k)$-fuzzy subalgebras of X, and $\nu, \gamma \leq \mu$. The chains of $(\in \vee\, q_k)$-level subalgebras of ν and γ are, respectively, given by

$$[\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r}$$

and

$$[\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r}.$$

Therefore ν and γ are non-equivalent and clearly $\mu = \nu \cup \gamma$. This completes the proof. □

Acknowledgements

The author is very grateful to referees for their valuable comments and suggestions for improving this paper.

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